Introduction to QCD

Lectures 3 and 4

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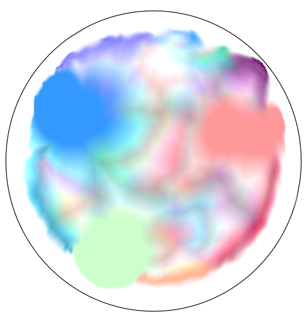
What do we know?

1



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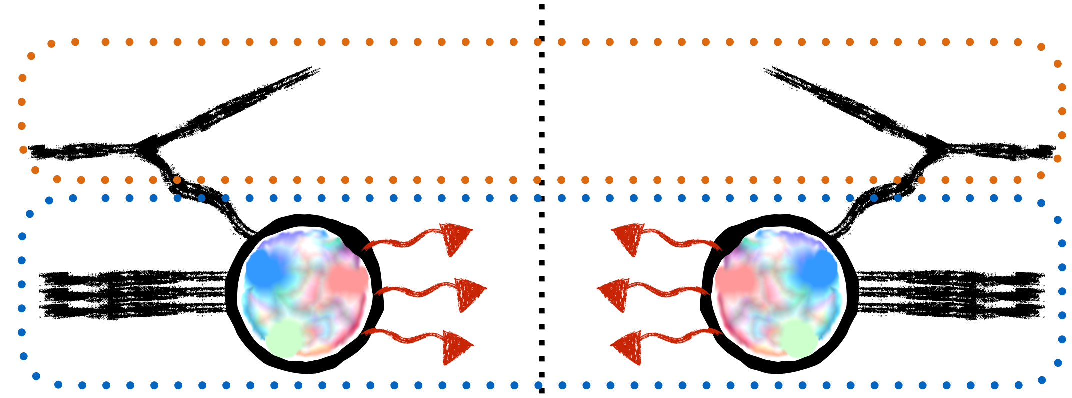
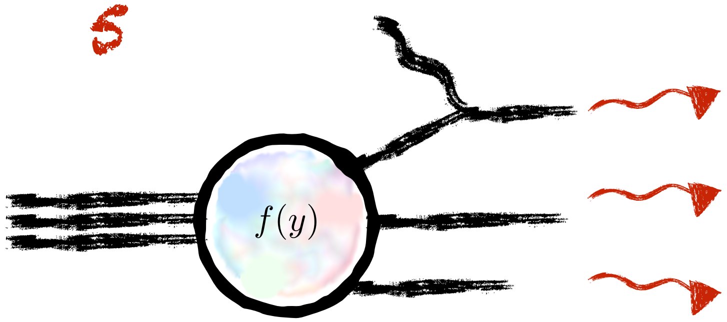


$$d\sigma = \frac{1}{2(S-M^2)} e^4 \frac{1}{Q^4} L^{\mu\nu} 4\pi M W_{\mu\nu} \frac{d^3 l'}{(2\pi)^3 2E'}$$

4

$$W_{\mu\nu} = -\left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right) W_1(\nu, Q^2) + \frac{1}{M^2}\left(P_\mu - q_\mu \frac{P\cdot q}{q^2}\right)\left(P_\nu - q_\nu \frac{P\cdot q}{q^2}\right) W_2(\nu, Q^2)$$

5



$$W_{\mu\nu} = \frac{1}{4\pi M} \int_x^1 \frac{dy}{y} f(y) \tilde{W}_{\mu\nu}$$

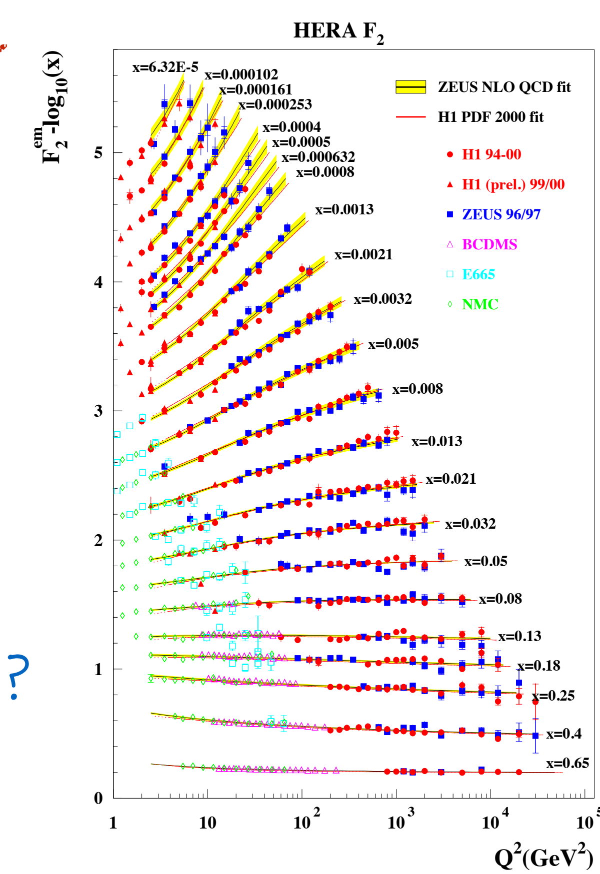
6

7

$$W_1(\nu, Q^2) = \sum_i \frac{Q_i^2}{2M} f_i(x)$$

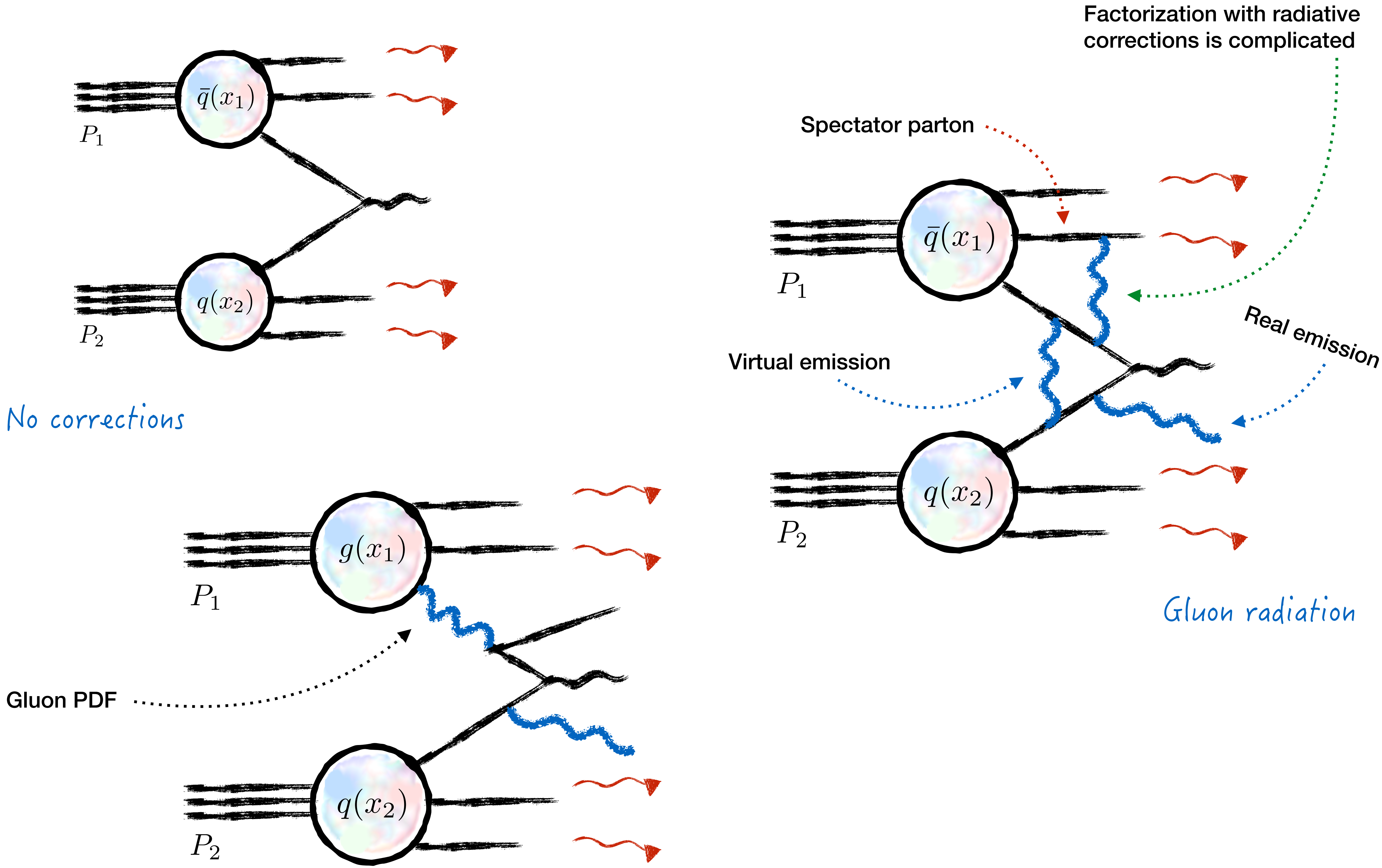
$$W_2(\nu, Q^2) = \sum_i \frac{Q_i^2}{\nu} x f_i(x)$$

8



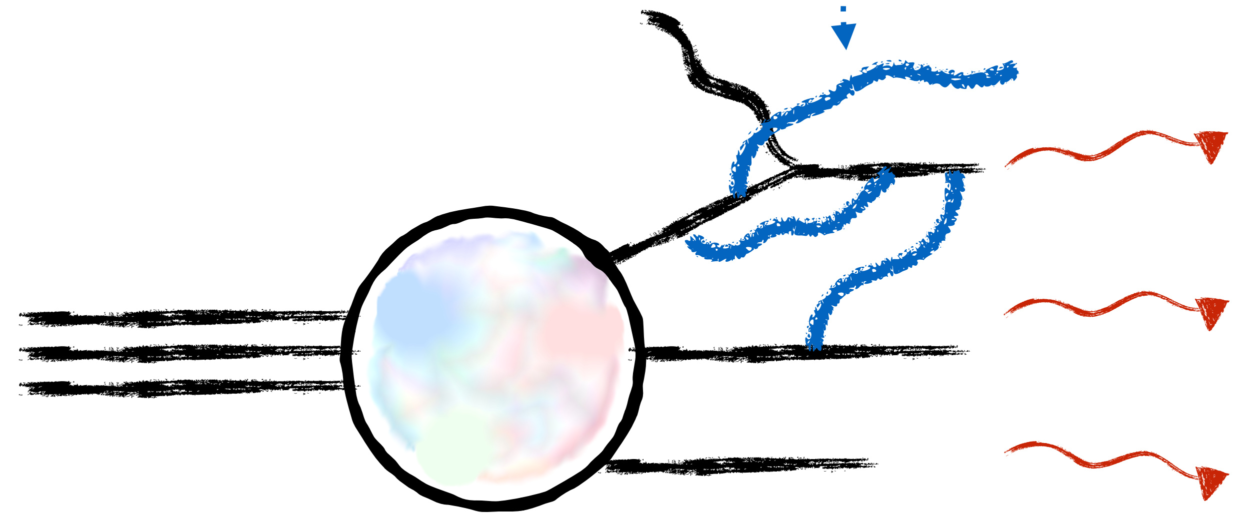
Where is Q?

Radiative corrections

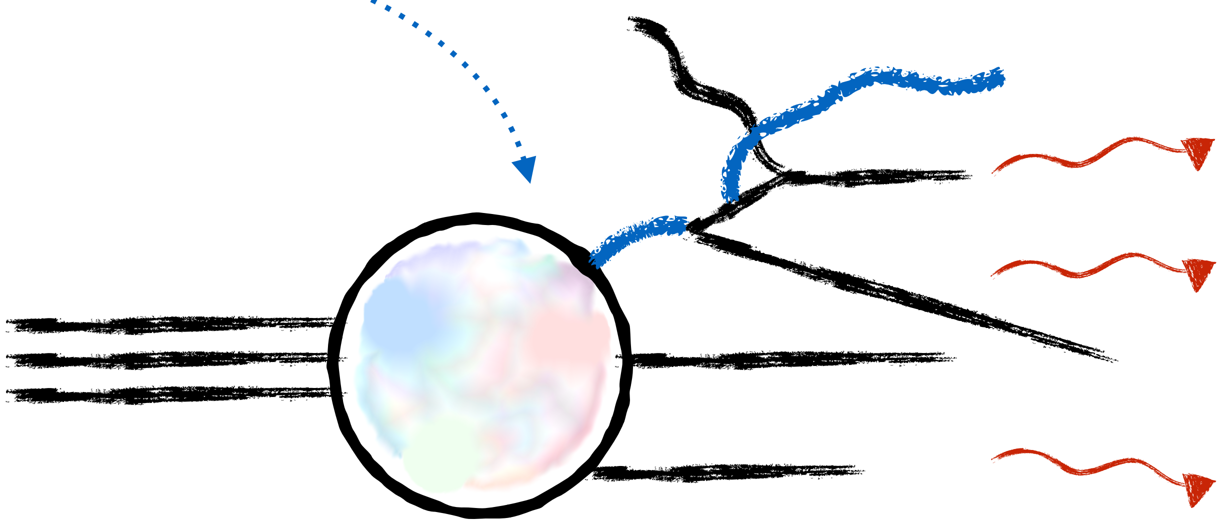


Radiative corrections

Gluons



Gluon distribution function

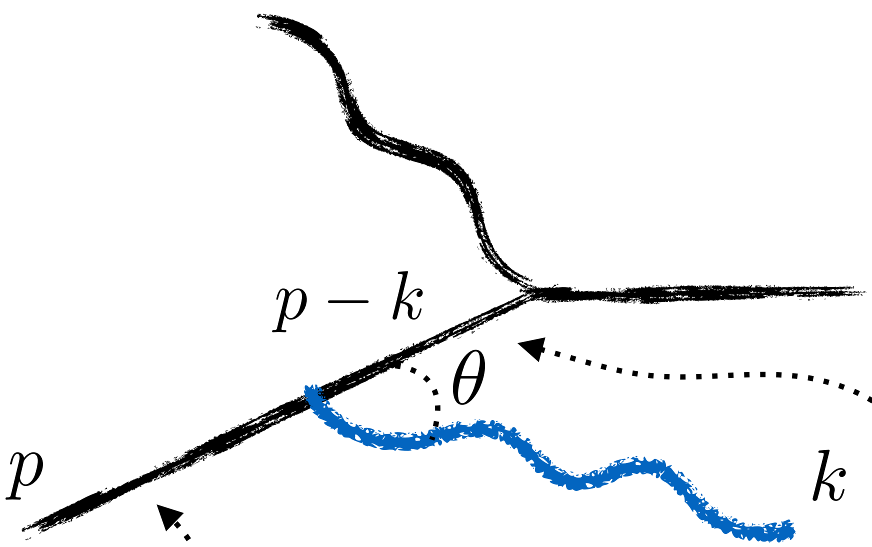


$$W_{\mu\nu} = \frac{1}{4\pi M} \int_x^1 \frac{dy}{y} f(y) \tilde{W}_{\mu\nu}$$

Should calculate corrections to both parts

There are large logarithms in this diagrams

Collinear singularity



This quark comes from the parton distribution

Quark propagator:

$$\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon} = -\frac{i(\not{p}-\not{k}+m)}{2p\cdot k}$$

Substitute

Let's choose a particular frame:

$$p = \left(E_p,\ 0,\ 0,\ E_p\sqrt{1-\frac{m^2}{E_p^2}}\right) \qquad k = (E_k, 0, E_k\sin\theta, E_k\cos\theta)$$

Quark velocity

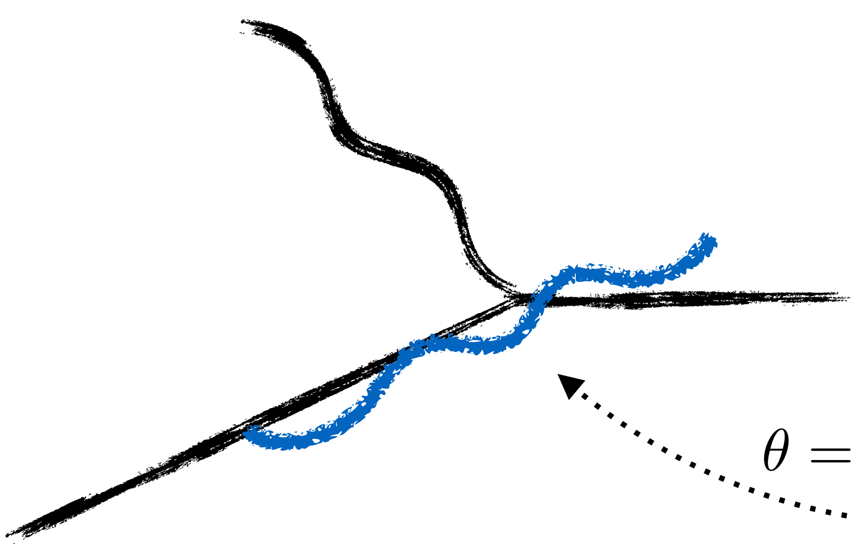
$$\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon} = -\frac{i(\not{p}-\not{k}+m)}{2E_pE_k\left\{1-\sqrt{1-\frac{m^2}{E_p^2}}\cos\theta\right\}}$$

If the quark mass is non-zero everything is fine

$m^2 = 0$

Quark mass regulates collinear singularity

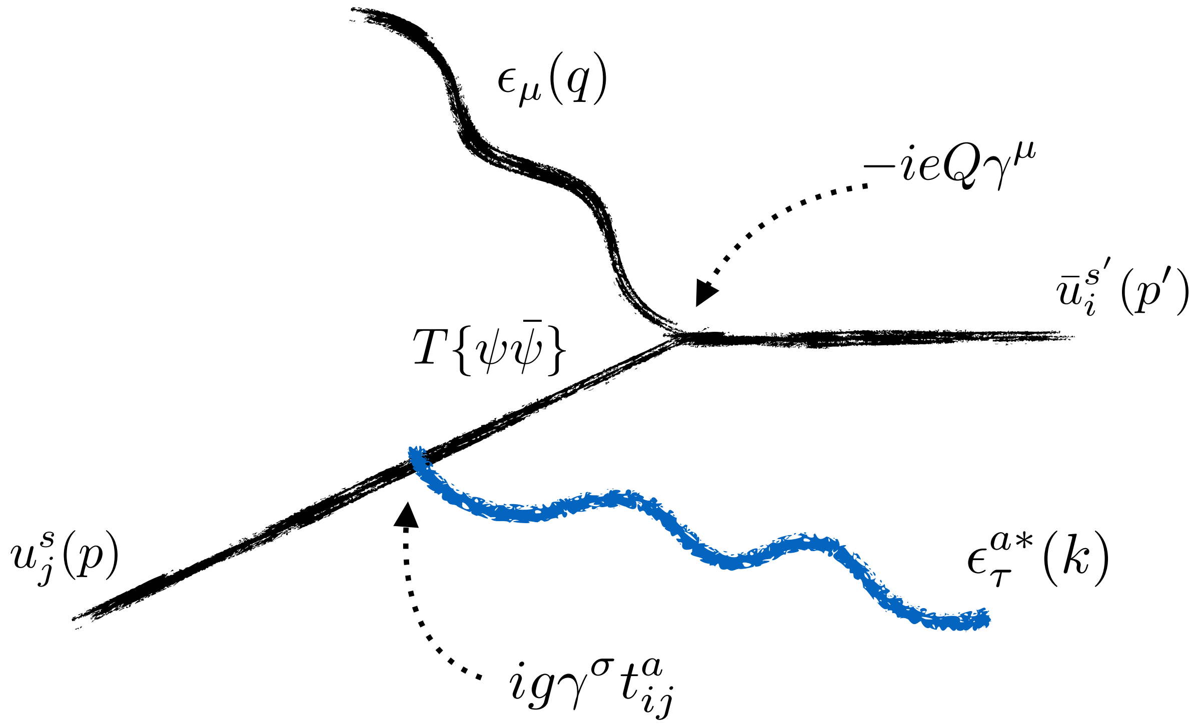
$$\frac{i(\not{p}-\not{k}+m)}{(p-k)^2-m^2+i\epsilon} = -\frac{i(\not{p}-\not{k}+m)}{2E_pE_k\left(1-\cos\theta\right)}$$



$\theta = 0$

At zero angle we get "collinear singularity" of the quark propagator

Large logarithm



$T\{\psi\bar\psi\}$

$$iM = -ieQ_i\bar u^{s'}(p')\gamma^\mu\psi \times ig\bar\psi\gamma^\sigma t^a_{ij}u^s(p) \times \epsilon_\mu(q)\epsilon^{a*}_\tau(k)$$

Write the contraction explicitly:

$$iM = eQ_ig\bar u^{s'}(p')\gamma^\mu \frac{i(\not p - \not k + m)}{(p-k)^2 - m^2 + i\epsilon}\gamma^\tau t^a_{ij}u^s(p) \times \epsilon_\mu(q)\epsilon^{a*}_\tau(k)$$

Complex conjugated amplitude

$$\bar u \equiv u^\dagger\gamma^0$$

$$(\gamma^\mu)^\dagger = \gamma^0\gamma^\mu\gamma^0$$

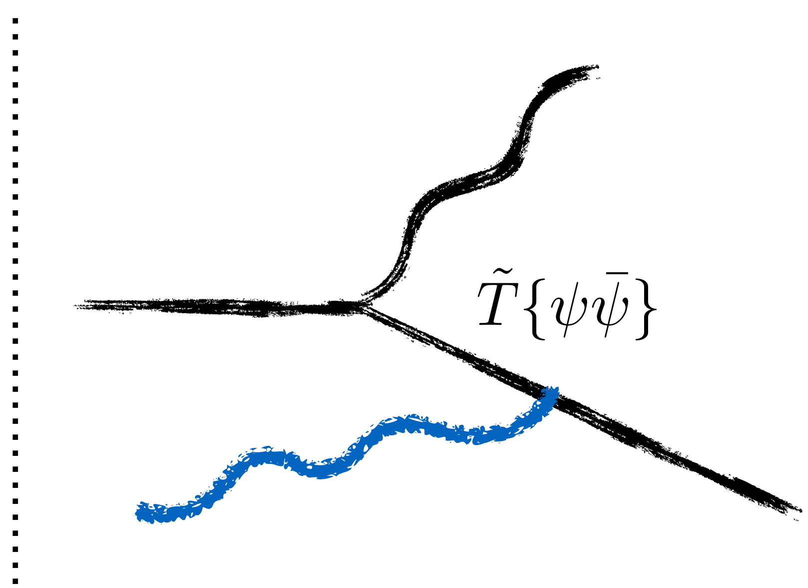
You will need

$$(\gamma^0)^2 = 1$$

$$-iM^* = eQ_ig\bar u^s(p)t^a_{ji}\gamma^\sigma \frac{-i(\not p - \not k + m)}{(p-k)^2 - m^2 - i\epsilon}\gamma^\nu u^{s'}(p') \times \epsilon^*_\nu(q)\epsilon^a_\sigma(k)$$

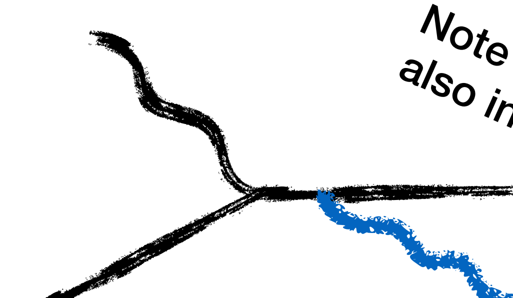
$\tilde T\{\psi\bar\psi\}$

Use the same color indexes. True after summation over polarization.

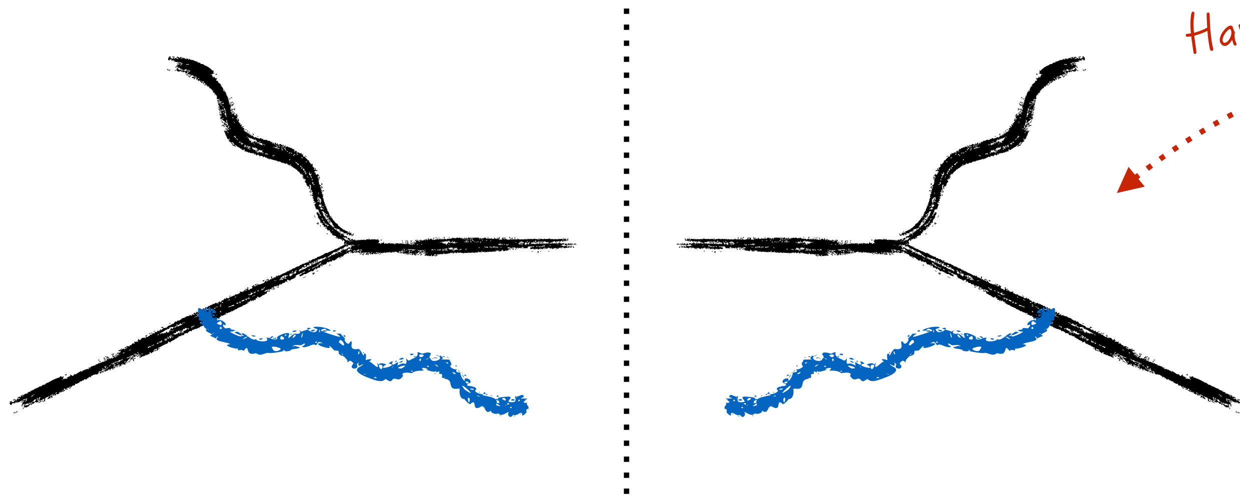


“Propagator” for the complex conjugated amplitude

Note that one should also include



Large logarithm



Have to take a lot of traces

The full expression is complicated.
That is typical for QCD.

The problem of any QCD calculation is to separate of what is important from what is not

We can estimate

$$|M|^2 \sim \frac{1}{2p\cdot k} \propto \frac{1}{2E_pE_k\left\{1-\sqrt{1-\frac{m^2}{E_p^2}}\cos\theta\right\}}$$

Jacobian

$$\tilde{\sigma} \sim \int \frac{d^3k}{(2\pi)^3 2E_k}|M|^2 \sim \int_0^{\pi} d\theta \frac{\sin\theta}{1-\sqrt{1-\frac{m^2}{E_p^2}}\cos\theta}$$

It is just an estimation which helps us to understand the general structure

$$\tilde{\sigma} \sim \ln\frac{E_p^2}{m^2} \sim \ln\frac{Q^2}{m^2}$$

At fixed Q is regulated by mass

The mass singularity can be found in all orders of perturbation theory

Non-perturbative problem

Collinear logarithm

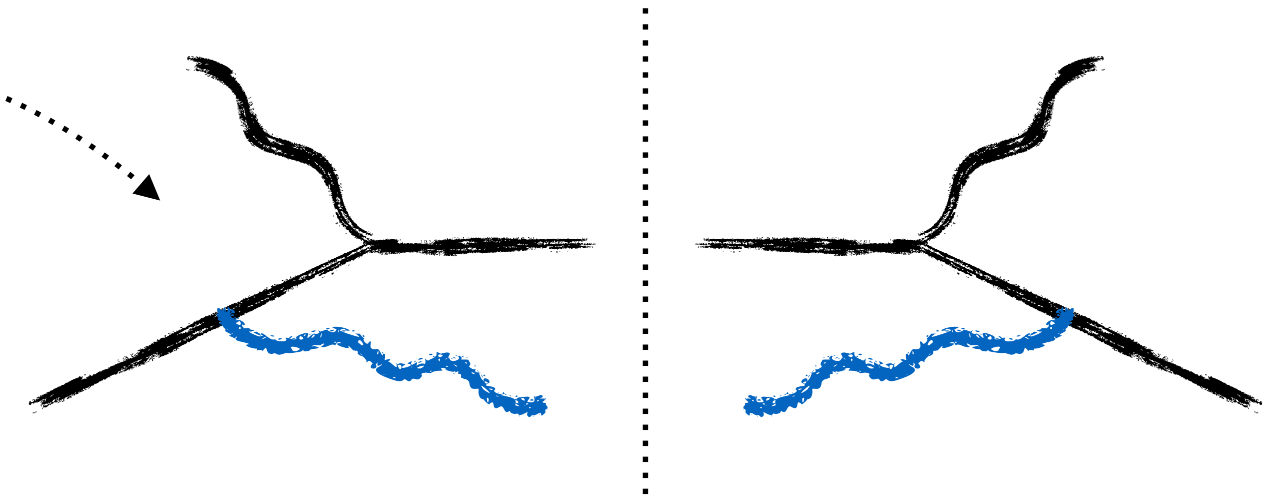
$$\alpha_s(Q^2)\ln\frac{Q^2}{m^2} \sim 1$$

We should sum this structure in all orders of perturbation theory

Cancellation of collinear divergence

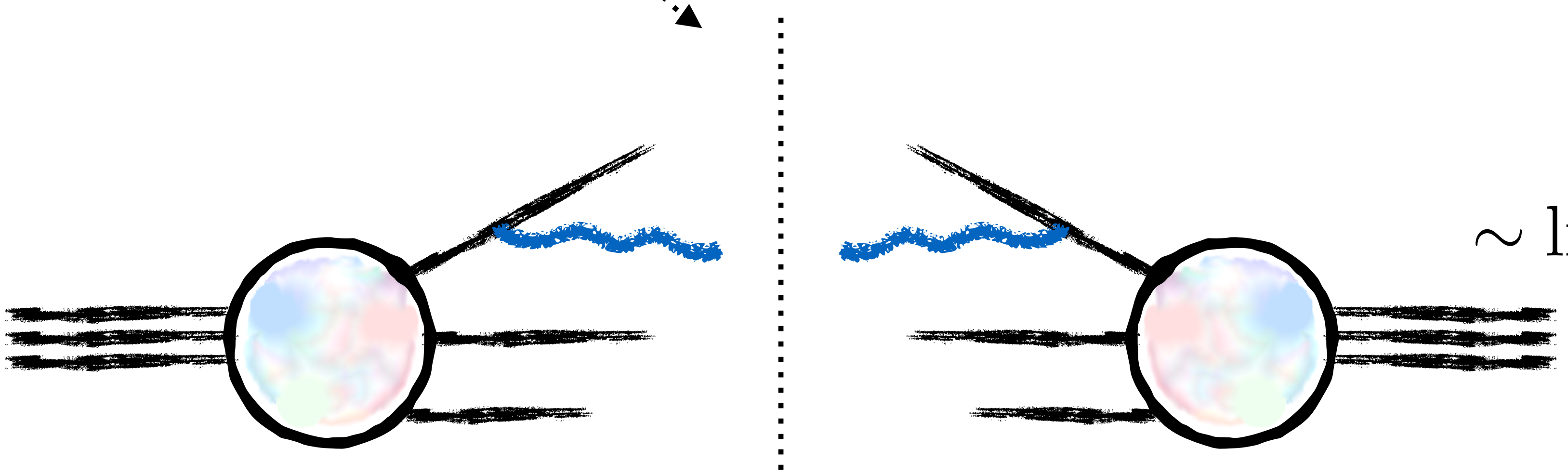
$$W_{\mu\nu} = \frac{1}{4\pi M}\int_x^1 \frac{dy}{y} f(y)\tilde{W}_{\mu\nu}$$

Partonic correction



$\sim \ln m^2$

PDF correction



$\sim \ln m^2$

They'll cancel in the hadronic cross section

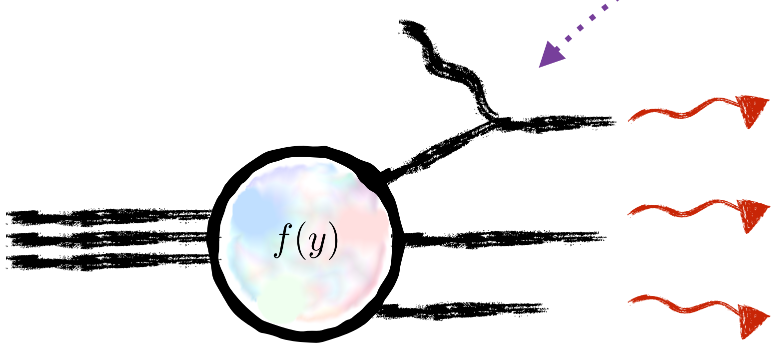
We will use another regulator

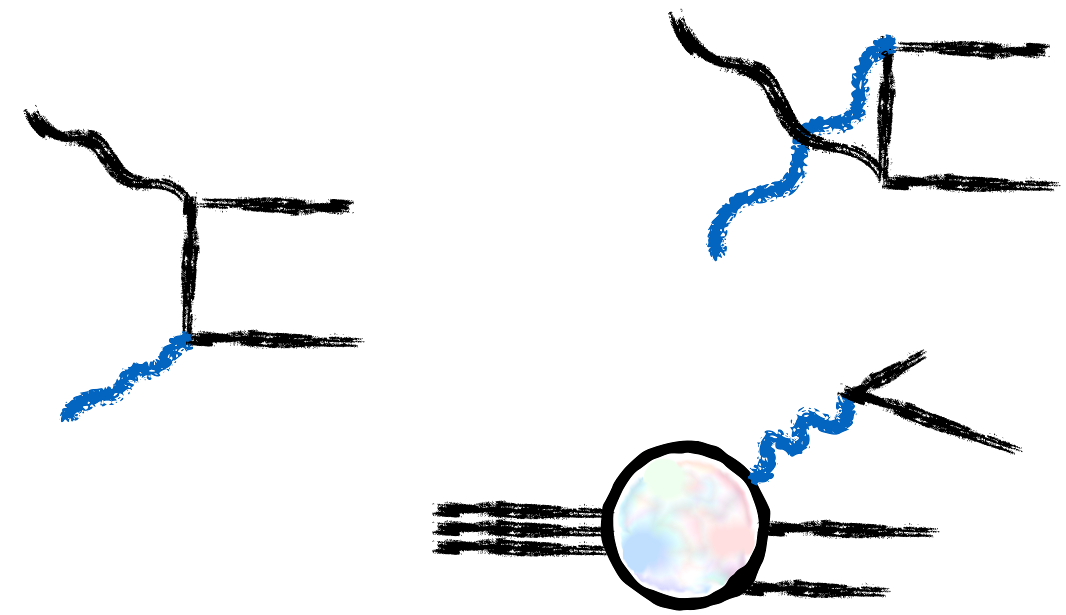
Dimensional regularization

Zero quark mass

Radiative corrections in DIS

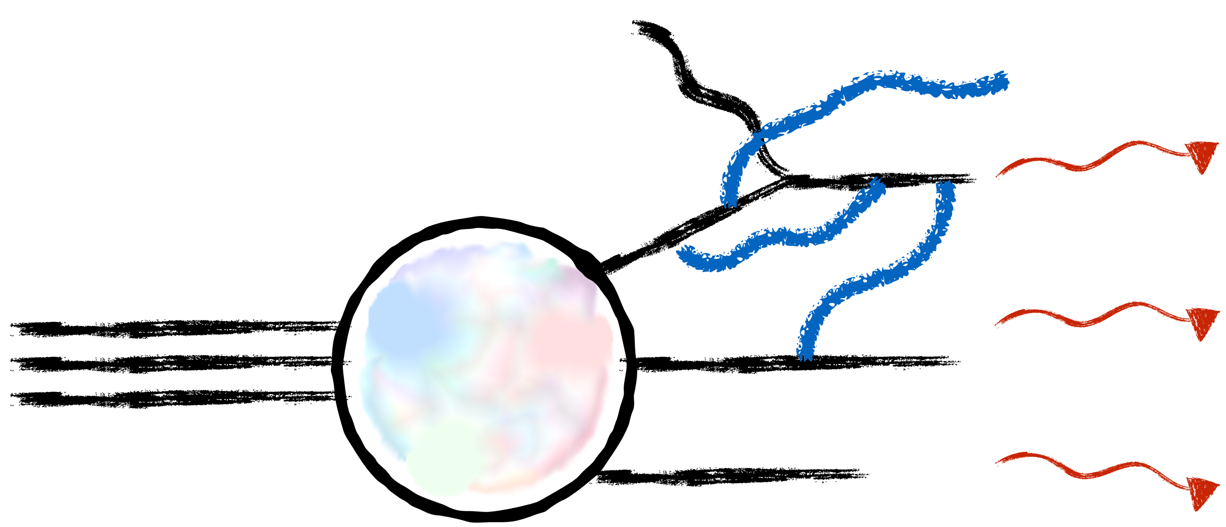
Corrections to the partonic cross section:





Corrections to distribution functions:

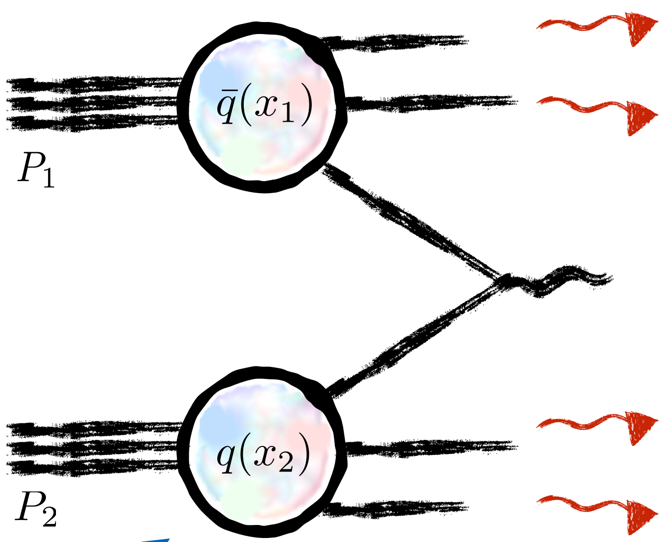
Gluon correction

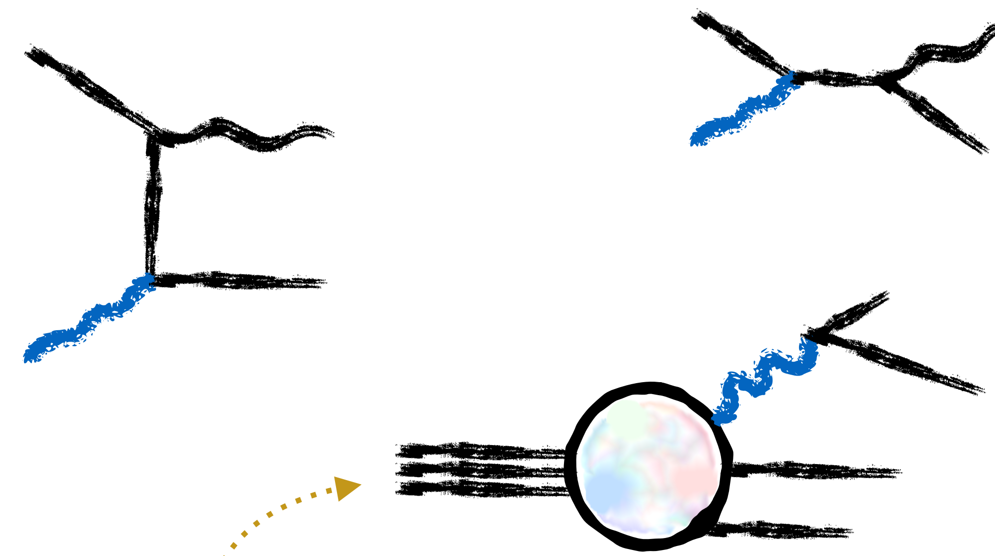


Gluon radiation

Radiative corrections in Drell-Yan

Corrections to the partonic cross section:

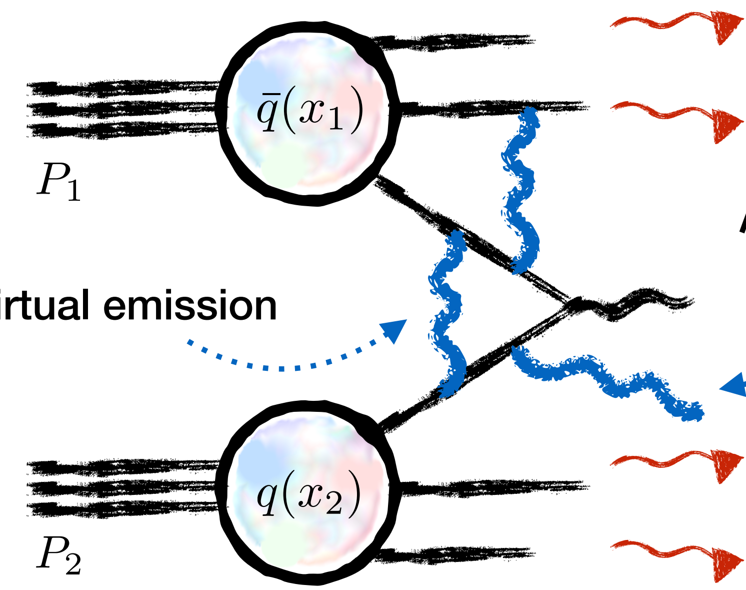




We have the same correction in DIS

Gluon correction

Corrections to distribution functions:



Virtual emission

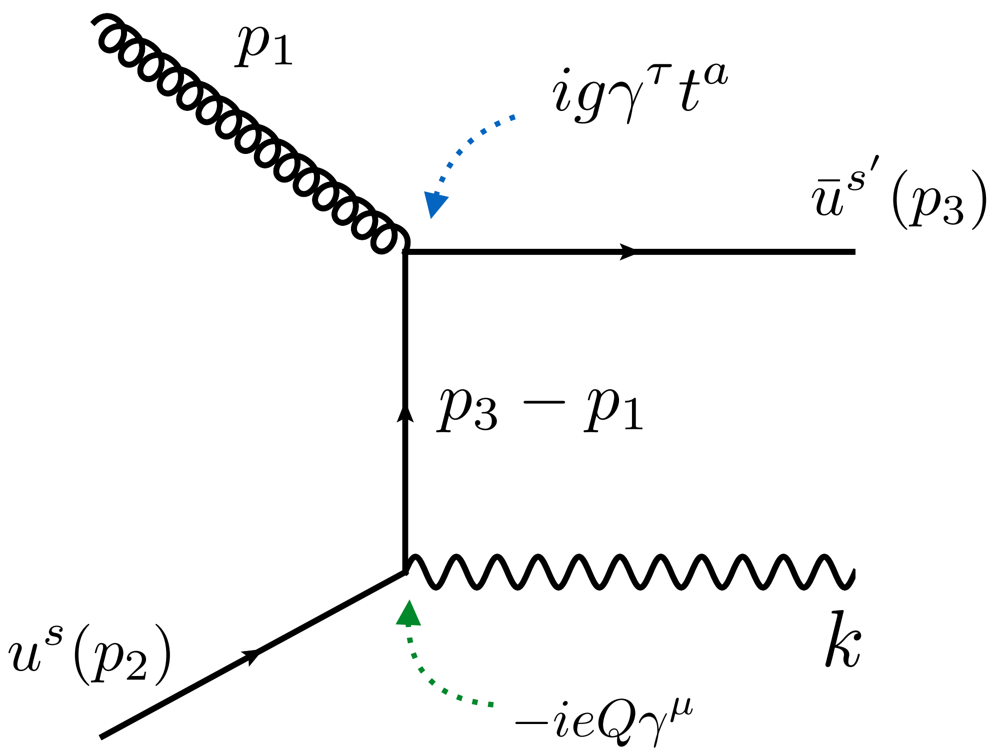
Real emission

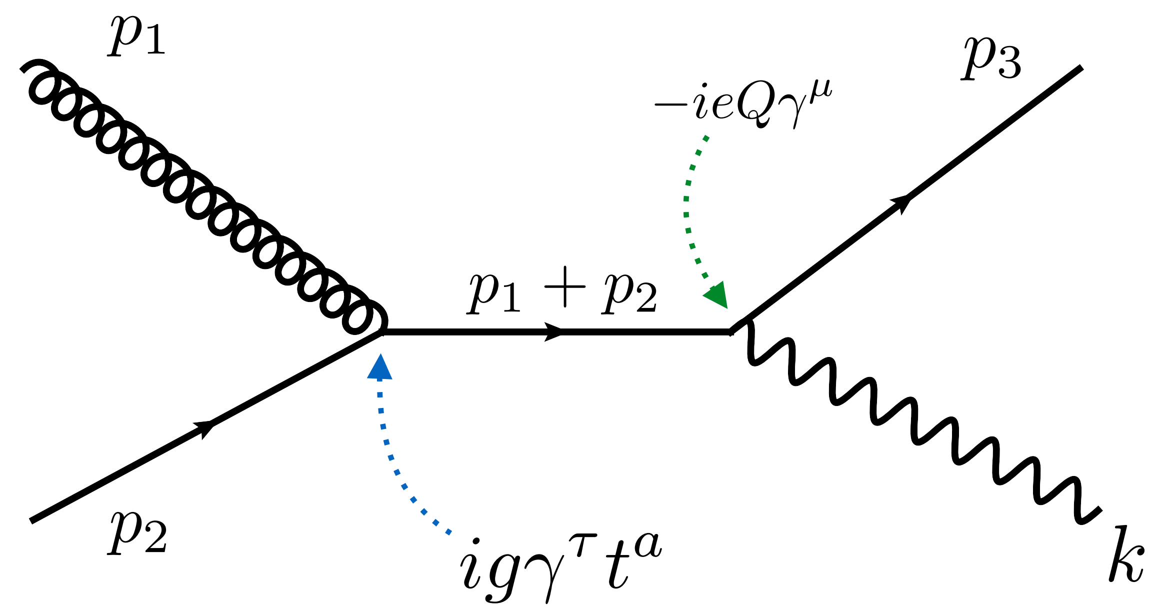
Gluon radiation

G. Altarelli, R.K. Ellis, and G. Martinelli,
Nucl. Phys. B157, 461 (1979)

Drell-Yan I: gluon distribution function

$k^2 \equiv Q^2$



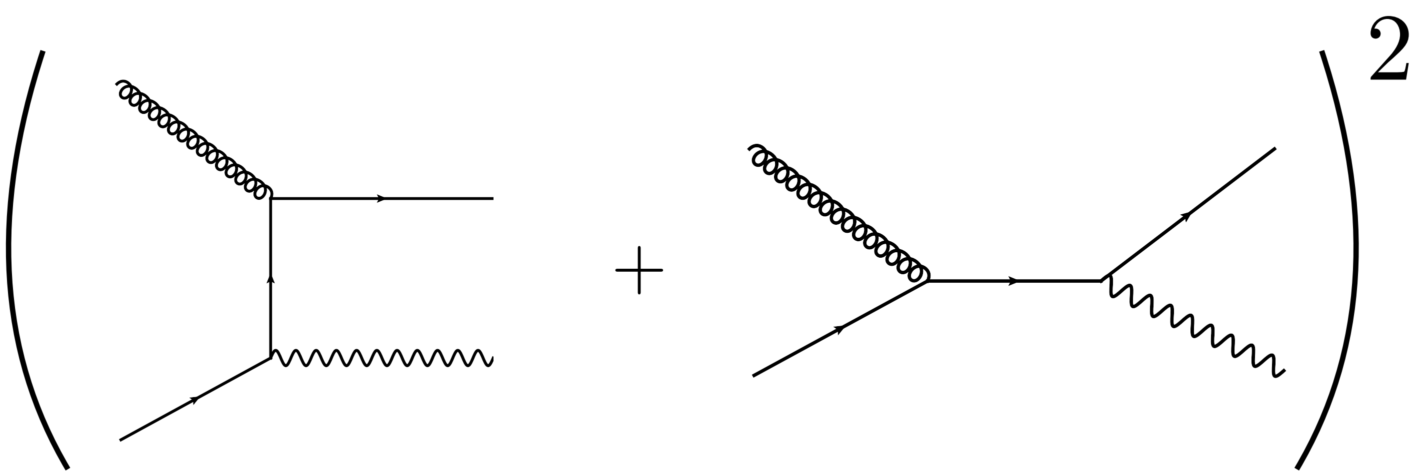


$$iM = eQ_i g \bar{u}^{s'}(p_3)\left\{\gamma^\tau t^a \frac{i}{\not{p}_3 - \not{p}_1}\gamma^\mu + \gamma^\mu \frac{i}{\not{p}_1 + \not{p}_2}\gamma^\tau t^a\right\} u^s(p_2) \times \epsilon^*_\mu(k)\epsilon^a_\tau(p_1)$$

Take the product

$$-iM^* = eQ_i g \bar{u}^{s}(p_2)\left\{\gamma^\nu \frac{-i}{\not{p}_3 - \not{p}_1} t^b\gamma^\sigma + t^b\gamma^\sigma \frac{-i}{\not{p}_1 + \not{p}_2}\gamma^\nu\right\} u^{s'}(p_3) \times \epsilon_\nu(k)\epsilon^{*b}_\sigma(p_1)$$

Drell-Yan: gluon distribution function



$$\sum_{\text{polarization}} \epsilon^{a}{}_{\tau}\epsilon^{*b}_{\sigma} \to -g_{\tau\sigma}\delta^{ab}$$

$$\sum_{s} u^s_i(p)\bar{u}^s_j(p) = (\not{p} + m)\delta_{ij}$$

$$Tr\{\gamma^\mu\gamma^\tau \dots \gamma^\sigma\} = Tr\{\gamma^\sigma \dots \gamma^\tau\gamma^\mu\}$$

$$|M|^2 = e^2Q_i^2g^2Tr\{t^at^a\} \times Tr\Big\{\not{p}_3\gamma^\tau(\not{p}_3 - \not{p}_1)\gamma^\mu\not{p}_2\gamma_\mu(\not{p}_3 - \not{p}_1)\gamma_\tau\frac{1}{t^2}$$
$$+2\not{p}_3\gamma^\tau(\not{p}_3 - \not{p}_1)\gamma^\mu\not{p}_2\gamma_\tau(\not{p}_1 + \not{p}_2)\gamma_\mu\frac{1}{st}$$
$$+\not{p}_3\gamma^\mu(\not{p}_1 + \not{p}_2)\gamma^\tau\not{p}_2\gamma_\tau(\not{p}_1 + \not{p}_2)\gamma_\mu\frac{1}{s^2}\Big\}$$

Color trace

Trace of gamma matrices

Collinear divergence is already here

We regulate it with dimensional regularization

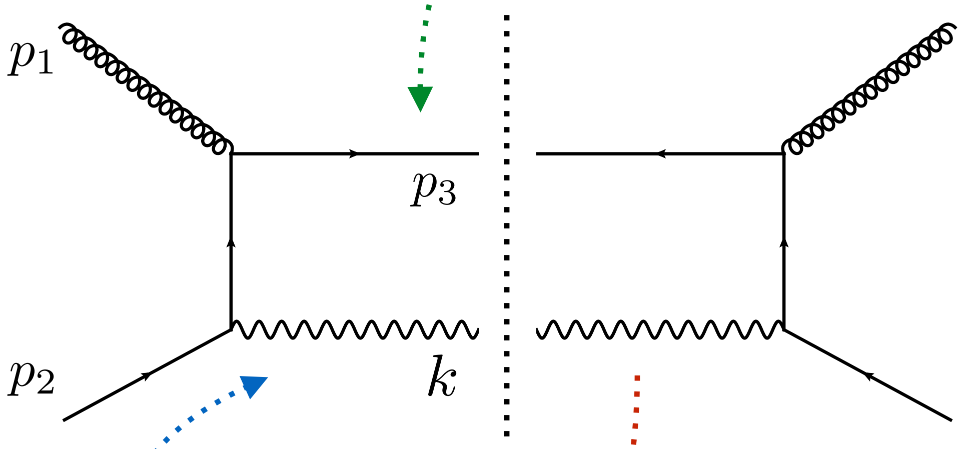
Mandelstam variables:

$$s = (p_1 + p_2)^2 = 2p_1 \cdot p_2 \qquad t = (p_1 - p_3)^2 = -2p_1 \cdot p_3 \qquad u = (p_2 - p_3)^2 = -2p_2 \cdot p_3$$

Dimensional regularization

$$d \equiv 4 - 2\epsilon \qquad \int d^4 p_3 \to \int d^{4-2\epsilon} p_3$$

We slightly change dimension



Cross section

This operation has “serious” consequences

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \qquad Tr\{\not{a}\not{b}\} = 4a \cdot b$$

$$g^{\mu\nu} g_{\mu\nu} = d \qquad \gamma^\mu \gamma_\mu = d$$

$$\gamma^\mu \not{a} \gamma_\mu = -2(1-\epsilon)\not{a} \qquad \gamma^\mu \not{a}\not{b} \gamma_\mu = 4a \cdot b - 2\epsilon \not{a}\not{b}$$

$$\gamma^\mu \not{a}\not{b}\not{c} \gamma_\mu = -2\not{c}\not{b}\not{a} + 2\epsilon \not{a}\not{b}\not{c}$$ **but** $$Tr\mathbb{1} = 4$$

$$Tr\{\gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma\} = 4\big(g^{\mu\nu} g^{\rho\sigma} - g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}\big)$$

$$\epsilon \to 0$$

Removes regularization

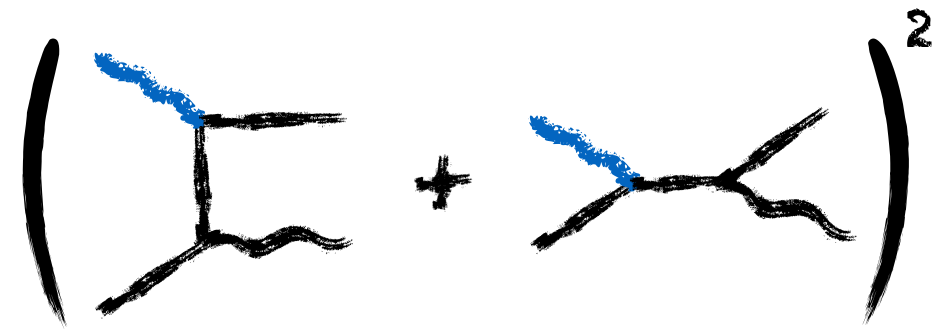
$$Tr\Big\{\not{p}_3 \gamma^\tau (\not{p}_3 - \not{p}_1)\gamma^\mu \not{p}_2 \gamma_\mu (\not{p}_3 - \not{p}_1)\gamma_\tau\Big\} = 4(1-\epsilon)^2 Tr\Big\{\not{p}_3(\not{p}_3 - \not{p}_1)\not{p}_2(\not{p}_3 - \not{p}_1)\Big\}$$

$$Tr\Big\{\not{p}_3 \gamma^\tau (\not{p}_3 - \not{p}_1)\gamma^\mu \not{p}_2 \gamma_\mu (\not{p}_3 - \not{p}_1)\gamma_\tau\Big\} = 32(1-\epsilon)^2 p_1 \cdot p_3 \; p_1 \cdot p_2 = -8(1-\epsilon)^2 st$$

$$Tr\{\not{p}_3 \gamma^\tau (\not{p}_3 - \not{p}_1)\gamma^\mu \not{p}_2 \gamma_\tau (\not{p}_1 + \not{p}_2)\gamma_\mu\} = 8(1-\epsilon)\Big\{-u(s+t+u) + \epsilon st\Big\} = 8(1-\epsilon)\Big\{-uM^2 + \epsilon st\Big\}$$

$$Tr\Big\{\not{p}_3 \gamma^\mu (\not{p}_1 + \not{p}_2)\gamma^\tau \not{p}_2 \gamma_\tau (\not{p}_1 + \not{p}_2)\gamma_\mu\Big\} = -8(1-\epsilon)^2 st$$

Drell-Yan: collinear divergence



$$\sum_{\text{spin, color}} |M|^2 = e^2 Q_i^2 g^2 Tr\{t^a t^a\} \times Tr\Big\{\not{p}_3 \gamma^\tau (\not{p}_3 - \not{p}_1)\gamma^\mu \not{p}_2 \gamma_\mu (\not{p}_3 - \not{p}_1)\gamma_\tau \frac{1}{t^2}$$
$$+2\not{p}_3 \gamma^\tau (\not{p}_3 - \not{p}_1)\gamma^\mu \not{p}_2 \gamma_\tau (\not{p}_1 + \not{p}_2)\gamma_\mu \frac{1}{st}$$
$$+\not{p}_3 \gamma^\mu (\not{p}_1 + \not{p}_2)\gamma^\tau \not{p}_2 \gamma_\tau (\not{p}_1 + \not{p}_2)\gamma_\mu \frac{1}{s^2}\Big\}$$

$$Tr\Big\{\not{p}_3 \gamma^\mu (\not{p}_1 + \not{p}_2)\gamma^\tau \not{p}_2 \gamma_\tau (\not{p}_1 + \not{p}_2)\gamma_\mu\Big\} = -8(1-\epsilon)^2 st$$

$$Tr\Big\{\not{p}_3 \gamma^\tau (\not{p}_3 - \not{p}_1)\gamma^\mu \not{p}_2 \gamma_\mu (\not{p}_3 - \not{p}_1)\gamma_\tau\Big\} = -8(1-\epsilon)^2 st$$

$$Tr\{\not{p}_3 \gamma^\tau (\not{p}_3 - \not{p}_1)\gamma^\mu \not{p}_2 \gamma_\tau (\not{p}_1 + \not{p}_2)\gamma_\mu\} = 8(1-\epsilon)\Big\{-uM^2 + \epsilon st\Big\}$$

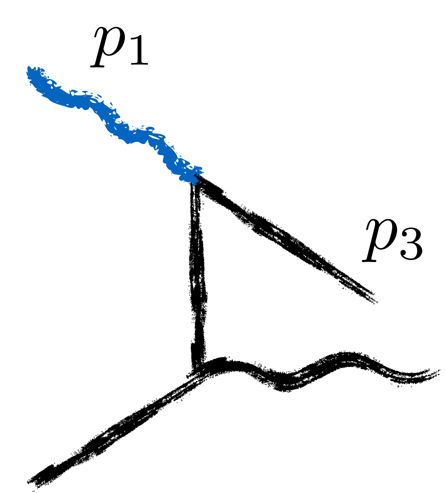
Result

$$\sum_{\text{spin, color}} |M|^2 = 8e^2 Q^2 g^2 Tr\{t^a t^a\}(1-\epsilon)\Big\{(1-\epsilon)\Big(\frac{s}{-t} + \frac{-t}{s}\Big) - 2\frac{uM^2}{st} + 2\epsilon\Big\}$$

Has divergence of the type

$$\frac{1}{t} \sim \frac{1}{p_1 \cdot p_3}$$

We've already seen this!



Drell-Yan: color

Eight matrices

$$\psi_i \to (\delta_{ij} + i\alpha^b t^b_{ij})\psi_j$$

Three indexes

Eight matrices

$$F^a_{\mu\nu} \to (\delta_{ac} + i\alpha^b T^b_{ac})F^c_{\mu\nu}$$

Eight indexes

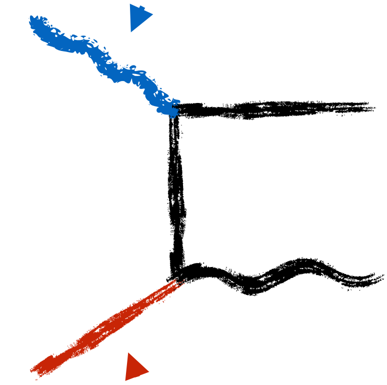
$$T^b_{ac} = if^{abc}$$

$$[t^a, t^b] = if^{abc}t^c$$

$$[T^a, T^b] = if^{abc}T^c$$

Representation of the same group

$\frac{1}{8}$ Adjoint representation



$\frac{1}{3}$ Fundamental representation

Average over color

$$tr[t^a] = 0 \qquad tr[T^a] = 0$$

$$tr[t^a, t^b] = \frac{1}{2}\delta^{ab} \qquad tr[T^a, T^b] = N_c\delta^{ab}$$

$$t^a_{ik}t^a_{kj} = \frac{N_c^2 - 1}{2N_c} \times \mathbb{1}_{ij} \qquad T^a_{be}T^a_{ec} = N_c \times \mathbb{1}_{bc}$$

We need this

$$|M|^2 \propto tr\{t^a, t^a\} = 4$$

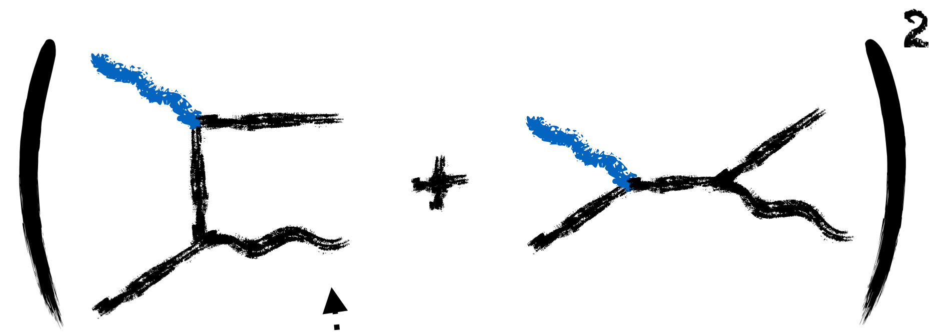
Drell-Yan: partonic cross section

Color

Gluon spin

Quark spin

$$\frac{1}{6}\frac{1}{d-2}\frac{1}{2}\sum_{\text{spin, color}}|M|^2 = \frac{1}{3}e^2Q_i^2g^2\left\{(1-\epsilon)\left(\frac{s}{-t}+\frac{-t}{s}\right)-2\frac{uM^2}{st}+2\epsilon\right\}$$



Parsonic cross-section:

$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{24}\sum_{\text{spin,color}}|M|^2\underbrace{\frac{d^3p_3}{(2\pi)^3 2E_3}\frac{d^3k}{(2\pi)^3 2E_k}(2\pi)^4\delta^4(p_1+p_2-p_3-k)}$$

Let's integrate over final phase-space

We live in the world with arbitrary number of dimensions:

$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{12(d-2)}\sum_{\text{spin,color}}|M|^2\underbrace{\frac{d^{d-1}p_3}{(2\pi)^{d-1}2E_3}\frac{d^{d-1}k}{(2\pi)^{d-1}2E_k}(2\pi)^d\delta^d(p_1+p_2-p_3-k)}$$

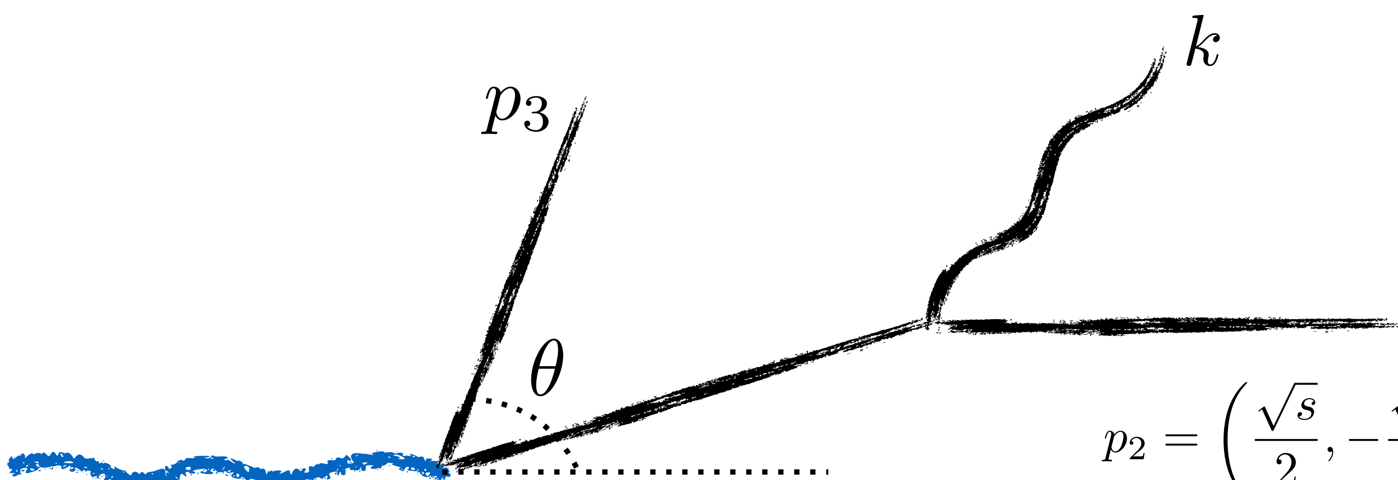
d-dimensional phase-space

Drell-Yan: phase space

$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{12(d-2)}\sum_{\text{spin,color}}|M|^2 \underbrace{\frac{d^{d-1}p_3}{(2\pi)^{d-1}2E_3}\frac{d^{d-1}k}{(2\pi)^{d-1}2E_k}(2\pi)^d\delta^d(p_1+p_2-p_3-k)}_{\text{d-dimensional phase-space}}$$

In this problem everything depends on angles, so phase integration is not trivial

Let's explicitly integrate in center of mass frame:



$$p_2 = \left(\frac{\sqrt{s}}{2}, -\frac{\sqrt{s}}{2}, 0_T\right)$$

$$p_1 = \left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s}}{2}, 0_T\right)$$

(d-2)-dimensional vector

$$v = \frac{1}{2}(1+\cos\theta)$$

This will be our future integration variable

Recall that:

$$s+t+u = M^2$$

By definition:

$$t = -2p_1^0 p_3^0(1-\cos\theta)$$

From momentum conservation:

$$p_3^0 = \sqrt{s} - k^0$$

We neglect quark mass:

$$k^0 = \frac{s}{2\sqrt{s}}\left(1+\frac{M^2}{s}\right)$$

$$t = -\frac{s}{2}\left(1-\frac{M^2}{s}\right)(1-\cos\theta)$$

$$t = -s\left(1-\frac{M^2}{s}\right)(1-v)$$

$$u = -s\left(1-\frac{M^2}{s}\right)v$$

Drell-Yan: phase space

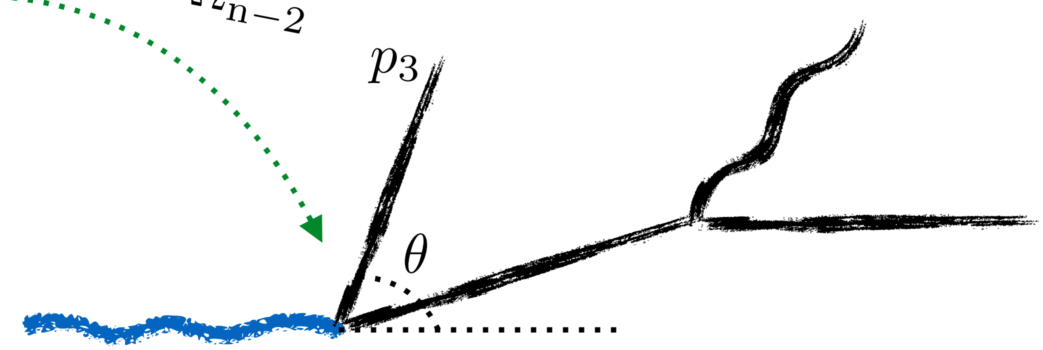
$$\int \frac{d^{d-1}p_3}{(2\pi)^{d-1}2E_3}\frac{d^{d-1}k}{(2\pi)^{d-1}2E_k}(2\pi)^d\delta^d(p_1+p_2-p_3-k) = \int \frac{d^{d-1}p_3}{(2\pi)^{d-1}2E_3}2\pi\delta(2p_1\cdot p_2 - 2p_1\cdot p_3 - 2p_2\cdot p_3 - M^2)$$

We need to perform this integral

We want to separate angles the amplitude doesn't depend on

$$\int_{-\infty}^{\infty} dk_1\cdots dk_{n-1}dk_n = \int_{-\infty}^{\infty} dk_n \underbrace{\int_0^{\infty} dk_{n-1}k_{n-1}^{n-2}\int d\Omega_{n-2}}_{n-1 \text{ dim space}}$$

symmetry in Ω_{n-2}



Polar coordinates

$$k_{n-1} = k\sin\theta,\;\; k_n = k\cos\theta$$

$$\int_{-\infty}^{\infty} dk_1\cdots dk_{n-1}dk_n = \int_0^{\infty} dk k^{n-1}\int_0^{\pi} d\theta \sin^{n-2}\theta \int d\Omega_{n-2}$$

$$\int d\Omega_{n-2} = 2^{n-2}\pi^{(n-2)/2}\frac{\Gamma(\frac{n-2}{2})}{\Gamma(n-2)}$$

We have this dependence in the amplitude

$$2\pi\delta(2p_1\cdot p_2 - 2p_1\cdot p_3 - 2p_2\cdot p_3 - M^2) = \frac{\pi}{\sqrt{s}}\delta(p_3 - \frac{s-M^2}{2\sqrt{s}})$$

$$v = \frac{1}{2}(1+\cos\theta)$$

Drell-Yan: phase space

$$\int \frac{d^{d-1}p_3}{(2\pi)^{d-1}2E_3} 2\pi\delta(2p_1\cdot p_2 - 2p_1\cdot p_3 - 2p_2\cdot p_3 - M^2) = \frac{1}{2(2\pi)^{d-1}}\int_0^\infty dp_3 p_3^{d-3}\int_0^\pi d\theta \sin^{d-3}\theta \int d\Omega_{d-3} \times \frac{\pi}{\sqrt{s}}\delta(p_3 - \frac{s-M^2}{2\sqrt{s}})$$

Dimension is shifted

$$d = 4 - 2\epsilon$$

$$\int_0^\pi d\theta \sin^{d-3}\theta = 2^{1-2\epsilon}\int_0^1 dv(1-v)^{-\epsilon}v^{-\epsilon}$$

$$\int d\Omega_{d-3} = 2^{1-2\epsilon}\pi^{\frac{1-2\epsilon}{2}}\frac{\Gamma(\frac{1-2\epsilon}{2})}{\Gamma(1-2\epsilon)} = \frac{2\pi^{1-\epsilon}}{\Gamma(1-\epsilon)}$$

Substitution yields:

$$\int \frac{d^{d-1}p_3}{(2\pi)^{d-1}2E_3}\frac{d^{d-1}k}{(2\pi)^{d-1}2E_k}(2\pi)^d\delta^d(p_1+p_2-p_3-k) = \frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^\epsilon\left(1-\frac{M^2}{s}\right)^{1-2\epsilon}\frac{1}{\Gamma(1-\epsilon)}\int_0^1 dv(1-v)^{-\epsilon}v^{-\epsilon}$$

We have taken almost all integrals in the phase space. The most difficult part is done!

The integral over angle we wish to calculate

Special functions

Beta function

$$B(\mu,\nu)=\int_0^1 dx x^{\mu-1}(1-x)^{\nu-1}$$

That is all you need to know about B-function

$$\int d\Omega_m = 2^m \pi^{m/2}\frac{\Gamma(m/2)}{\Gamma(m)}$$

Can prove by iteration

$$B(\mu,\nu)=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)} \qquad B(\mu,\nu)=2\int_0^{\pi/2} d\theta \sin^{2\mu-1}\theta \cos^{2\nu-1}\theta$$

Gamma function

$$\Gamma(z)=\int_0^\infty d\beta \beta^{z-1}\exp(-\beta)$$

Recursion relation

$$z\Gamma(z)=\Gamma(z+1)$$

Gamma function of the integer argument

$$\Gamma(n)=(n-1)! \qquad \Gamma(1/2)=\pi^{1/2}$$

Expansion relation

$$\Gamma(1+\epsilon)=1-\epsilon\gamma_E+\dots$$

$$\gamma_E \approx 0.54$$

$$\Gamma(n)\Gamma(1/2)=2^{n-1}\Gamma(n/2)\Gamma((n+1)/2)$$

Drell-Yan: partonic cross section

$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{12(d-2)}\sum_{\text{spin,color}}|M|^2 \frac{d^{d-1}p_3}{(2\pi)^{d-1}2E_3}\frac{d^{d-1}k}{(2\pi)^{d-1}2E_k}(2\pi)^d\delta^d(p_1+p_2-p_3-k)$$

Flux factor

$$\frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^{\epsilon}\left(1-\frac{M^2}{s}\right)^{1-2\epsilon}\frac{1}{\Gamma(1-\epsilon)}\int_0^1 dv(1-v)^{-\epsilon}v^{-\epsilon}$$

$$\frac{1}{6}\frac{1}{d-2}\frac{1}{2}\sum_{\text{spin, color}}|M|^2 = \frac{1}{3}e^2Q_i^2g^2\left\{(1-\epsilon)\left(\frac{s}{-t}+\frac{-t}{s}\right)-2\frac{uM^2}{st}+2\epsilon\right\}$$

$$u = -s\left(1-\frac{M^2}{s}\right)v$$

$$t = -s\left(1-\frac{M^2}{s}\right)(1-v)$$

Huge, but exact result!

$$\int^1 \frac{dv}{1-v} \sim \ln(1-v)\Big|^1$$

$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{3}e^2Q_i^2g^2\frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^{\epsilon}\left(1-\frac{M^2}{s}\right)^{1-2\epsilon}\frac{1}{\Gamma(1-\epsilon)}\times\int_0^1 dv(1-v)^{-\epsilon}v^{-\epsilon}$$

$$\times\left\{(1-\epsilon)\left(\frac{1}{(1-M^2/s)(1-v)}+\left(1-\frac{M^2}{s}\right)(1-v)\right)-2\frac{M^2}{s}\frac{v}{1-v}+2\epsilon\right\}$$

Without dimensional regularization the integral is explicitly divergent

The collinear divergence?

Drell-Yan: collinear divergence

$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{3}e^2Q_i^2g^2\frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^{\epsilon}\left(1-\frac{M^2}{s}\right)^{1-2\epsilon}\frac{1}{\Gamma(1-\epsilon)}\times\int_0^1 dv(1-v)^{-\epsilon}v^{-\epsilon}$$
$$\times\Big\{(1-\epsilon)\Big(\frac{1}{(1-M^2/s)(1-v)}+\left(1-\frac{M^2}{s}\right)(1-v)\Big)-2\frac{M^2}{s}\frac{v}{1-v}+2\epsilon\Big\}$$

Let's explicitly integrate over angle

Integral is trivial

$$B(\mu,\nu)\equiv\int_0^1 dx x^{\mu-1}(1-x)^{\nu-1}=\frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}$$

Do you recognize the structure?

$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{3}e^2Q_i^2g^2\frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^{\epsilon}\left(1-\frac{M^2}{s}\right)^{1-2\epsilon}\frac{1}{\Gamma(1-\epsilon)}$$
$$\times\left\{(1-\epsilon)\left(\frac{1}{1-M^2/s}\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}+\left(1-\frac{M^2}{s}\right)\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\right)-2\frac{M^2}{s}\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)}+2\epsilon\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\right\}$$

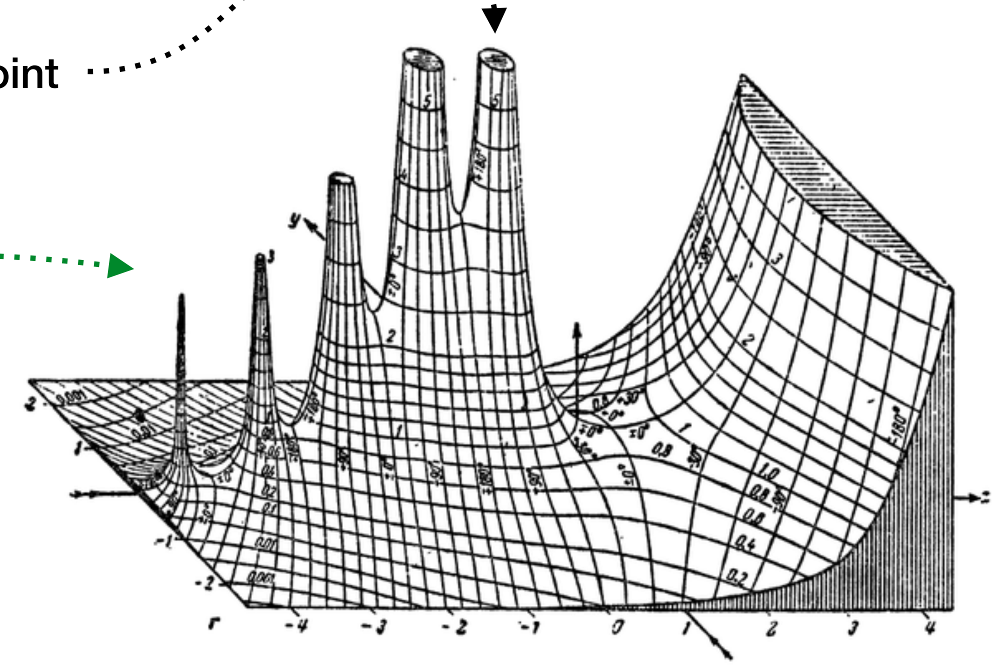
The result is finite at finite shift

Let's expand around this point

$$\Gamma(-\epsilon)$$

Collinear divergence is in this function

Poles of the gamma function



Drell-Yan: collinear divergence

$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{3}e^2Q_i^2g^2\frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^{\epsilon}\left(1-\frac{M^2}{s}\right)^{1-2\epsilon}\frac{1}{\Gamma(1-\epsilon)}$$

We want to study this result at $\epsilon \to 0$

$$\times\left\{(1-\epsilon)\left(\frac{1}{1-M^2/s}\frac{\Gamma(1-\epsilon)\Gamma(-\epsilon)}{\Gamma(1-2\epsilon)}+\left(1-\frac{M^2}{s}\right)\frac{\Gamma(1-\epsilon)\Gamma(2-\epsilon)}{\Gamma(3-2\epsilon)}\right)-2\frac{M^2}{s}\frac{\Gamma(2-\epsilon)\Gamma(-\epsilon)}{\Gamma(2-2\epsilon)}+2\epsilon\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\right\}$$

Apply

We need:

$$\Gamma(x+1) = x\Gamma(x) \qquad \Gamma(-\epsilon) = -\frac{1}{\epsilon}\Gamma(1-\epsilon)$$

$$\frac{1-\epsilon}{1-2\epsilon} \sim 1+\epsilon$$

$$\left\{-\frac{1}{\epsilon}\frac{1-\epsilon}{1-M^2/s}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}+\left(1-\frac{M^2}{s}\right)\frac{\Gamma^2(2-\epsilon)}{\Gamma(3-2\epsilon)}+\frac{2M^2}{\epsilon s}\frac{1-\epsilon}{1-2\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}+2\epsilon\frac{\Gamma^2(1-\epsilon)}{\Gamma(2-2\epsilon)}\right\}$$

Finite

Finite

$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{3}e^2Q_i^2g^2\frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^{\epsilon}\left(1-\frac{M^2}{s}\right)^{1-2\epsilon}\frac{1}{\Gamma(1-\epsilon)}$$

$$\times\left\{-\frac{1}{\epsilon}\frac{\Gamma^2(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{1-\epsilon}{1-M^2/s}-\frac{2M^2}{s}(1+\epsilon)\right]+\frac{1}{2}\left(1-\frac{M^2}{s}\right)+\dots\right\}$$

We have explicitly separated collinear divergence

Drell-Yan: final result

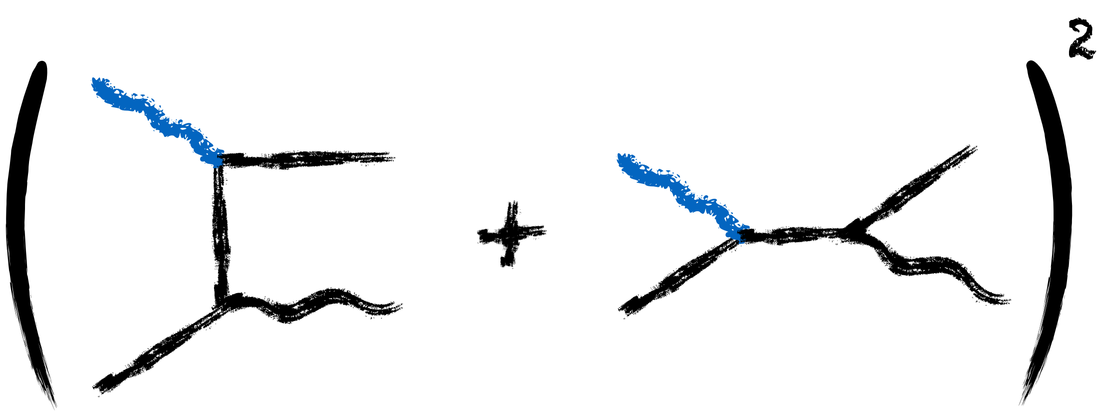
$$d\tilde{\sigma} = \frac{1}{2s}\frac{1}{3}e^2 Q_i^2 g^2 \frac{1}{8\pi}\left(\frac{4\pi}{M^2}\right)^{\epsilon}\left(\frac{M^2}{s}\right)^{\epsilon}\left(1-\frac{M^2}{s}\right)^{-2\epsilon}\times\left\{-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)}\left[\frac{M^4}{s^2}+\left(1-\frac{M^2}{s}\right)^2\right]+\frac{3}{2}+\frac{M^2}{s}-\frac{3}{2}\frac{M^4}{s^2}\right\}$$

Final expansion

$$a^{\epsilon} = e^{\epsilon \ln a} = 1 + \epsilon \ln a + \ldots$$

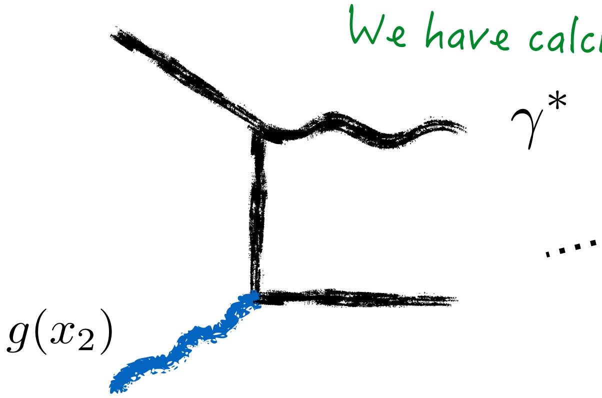
$$d\tilde{\sigma} = \frac{e^2 Q_i^2 g^2}{48\pi s}\times\left\{\left(-\frac{1}{\epsilon}\frac{\Gamma(1-\epsilon)}{\Gamma(1-2\epsilon)} - \ln\frac{4\pi}{M^2}\frac{M^2/s}{(1-M^2/s)^2}\right)\left[\frac{M^4}{s^2}+\left(1-\frac{M^2}{s}\right)^2\right]+\frac{3}{2}+\frac{M^2}{s}-\frac{3}{2}\frac{M^4}{s^2}\right\}$$

This is our final result



How to extract gluon?

We have calculated this

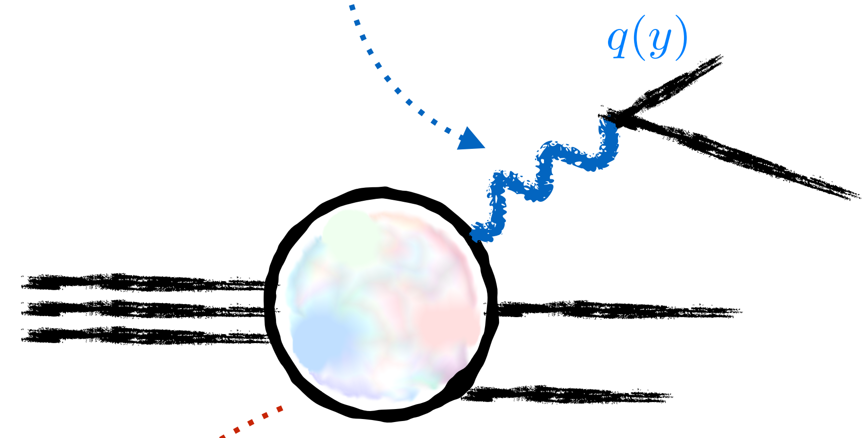


$$d\sigma = \int_{\tau_0}^{1} dx_1 \int_{\tau_0/x_1}^{1} dx_2 \bar{q}(x_1) g(x_2) d\tilde{\sigma}(q\bar{q} \to \gamma^*)$$

gluon distribution

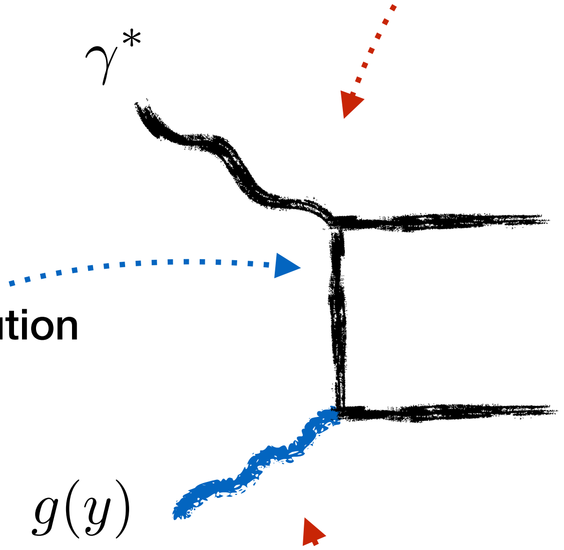
"free" gluon

Calculate some c.s. (form factor) and extract this correction



Correction to quak distribution

$$W_{\mu\nu} = \frac{1}{4\pi M} \int_x^1 \frac{dy}{y} q(y) \tilde{W}_{\mu\nu}$$



quark distribution

We can consider this as definition of distribution functions $q(y)$

Compare

$$F_2(x, Q^2) = \sum_i Q_i^2 x \left\{ q_i(x, Q^2) + \bar{q}_i(x, Q^2) \right\}$$

We calculate gluon corrections to this functions

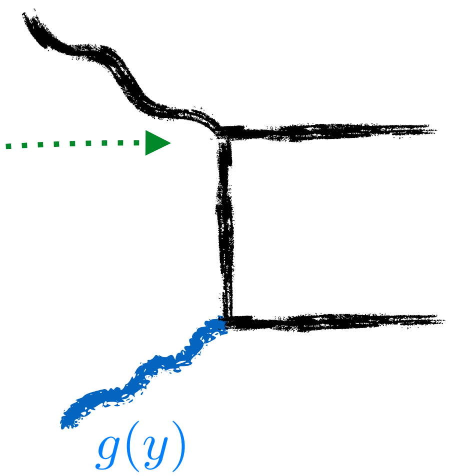
Calculate this diagram with the gluon distribution and compare with the form factor to extract quark distributions

Transverse and longitudinal structure functions

We want to calculate contribution of this diagram into the hadronic tensor **1**

$$W_{\mu\nu} = \frac{1}{4\pi M}\int_x^1 \frac{dy}{y} g(y) \tilde{W}_{\mu\nu}$$

Leading order



Compare with form factor **2**

$$F_2(x,Q^2) = \sum_i Q_i^2 x \left\{ q_i(x,Q^2) + \bar{q}_i(x,Q^2) \right\}$$

This will give gluon corrections to quark distribution functions **3**

It is much more convenient to calculate transverse and longitudinal structure functions

$$W_T \equiv -g^{\mu\nu} W_{\mu\nu}$$

$$W_L \equiv P^\mu P^\nu W_{\mu\nu}$$

Recall the general structure of the hadronic tensor

$$W_T = (3-2\epsilon) W_1 - \frac{\nu^2}{Q^2} W_2$$

$$W_L = -\frac{M^2\nu^2}{Q^2} W_1 + \frac{M^2\nu^4}{Q^4} W_2$$

Can solve this system

$$\nu = \frac{P\cdot q}{M}$$

$$Q^2 \gg M^2$$

Don't forget about "deep" regime

Transverse and longitudinal structure functions

$$W_T = (3-2\epsilon)W_1 - \frac{\nu^2}{Q^2}W_2$$

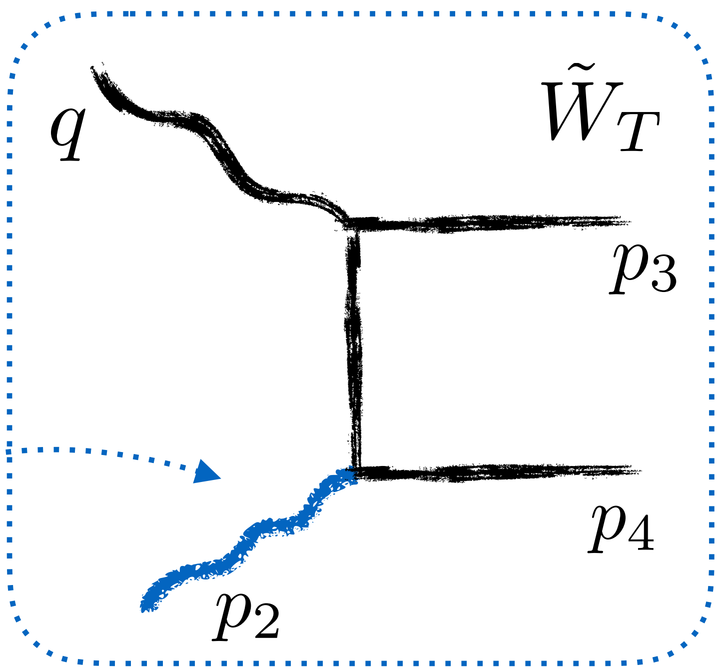
$$W_L = -\frac{M^2\nu^2}{Q^2}W_1 + \frac{M^2\nu^4}{Q^4}W_2$$

$F_2 = \nu W_2$

$$(1-\epsilon)\frac{1}{M}F_2 = xW_T + 4\frac{x^3}{Q^2}(3-2\epsilon)W_L$$

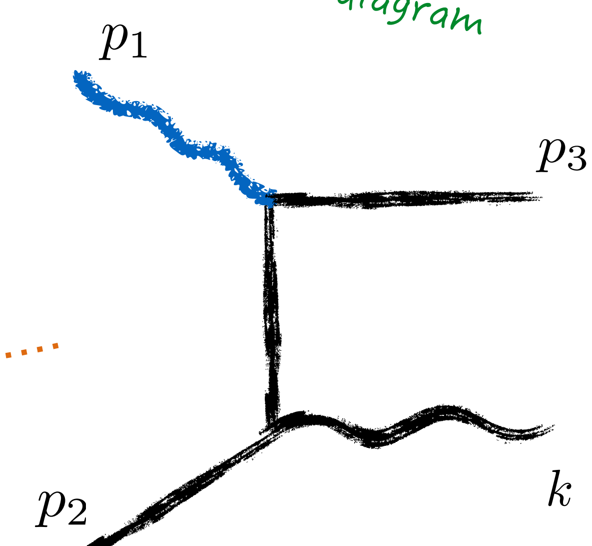
We have expressed form factor in terms of functions we want to calculate

$$W_T = \frac{1}{4\pi M}\int_x^1 \frac{dy}{y} g(y)\tilde{W}_T$$



"Free gluon"

We have already calculated this diagram



Crossing symmetry

$$s = (p_1+p_2)^2 \to (p_2-p_4)^2 = t$$

$$t = (p_1-p_3)^2 \to (p_2-p_3)^2 = u$$

$$u = (p_2-p_3)^2 \to (p_3+p_4)^2 = s$$

$$p_1 \to p_2$$

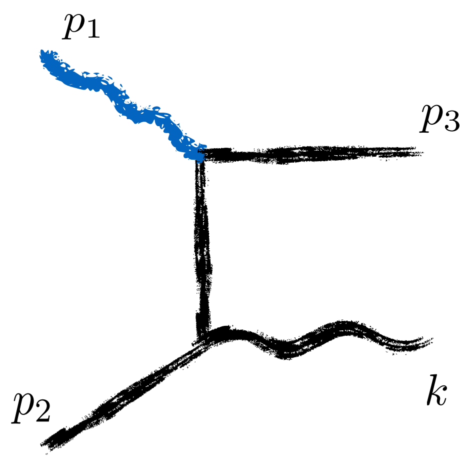
$$p_2 \to -p_4$$

$$p_3 \to p_3$$

$$k \to -q$$

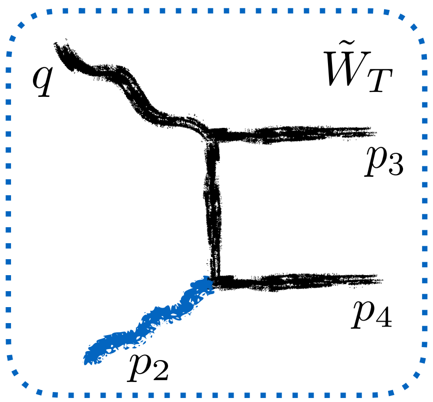
Scattering amplitude

We have already calculated this diagram



$$\sum_{\text{spin, color}} |M|^2 = 8e^2 Q_i^2 g^2 Tr\{t^a t^a\}(1-\epsilon)\left\{(1-\epsilon)\left(\frac{s}{-t}+\frac{-t}{s}\right) - 2\frac{uM^2}{st} + 2\epsilon\right\}$$

Crossing symmetry gives the result for this diagram



$$s = (p_1+p_2)^2 \to (p_2-p_4)^2 = t$$

$$t = (p_1-p_3)^2 \to (p_2-p_3)^2 = u$$

$$u = (p_2-p_3)^2 \to (p_3+p_4)^2 = s$$

$$Q^2 \to q^2$$

$$\sum_{\text{spin, color}} |M|^2 = -8e^2 Q_i^2 g^2 Tr\{t^a t^a\}(1-\epsilon)\left\{-(1-\epsilon)\left(\frac{t}{u}+\frac{u}{t}\right) - 2\frac{sq^2}{tu} + 2\epsilon\right\}$$

Fermion loop

$$W_T = \frac{1}{4\pi M}\int_x^1 \frac{dy}{y} g(y) \tilde{W}_T$$

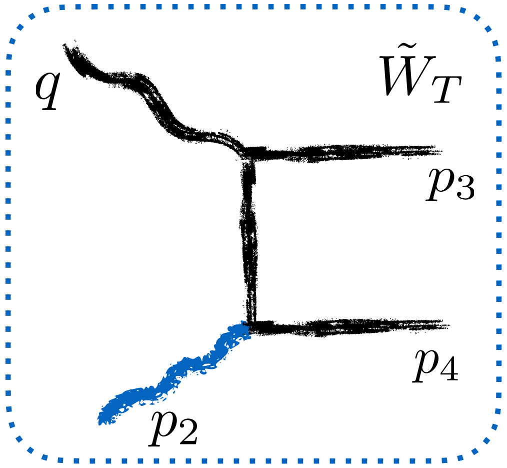
$$W_T \equiv -g^{\mu\nu} W_{\mu\nu}$$

Equivalent to summation over photon polarizations

+ average over gluon color and spin (the coefficient is different than before)

Phase space

$$\frac{1}{16(1-\epsilon)} \sum_{\text{spin, color}} |M|^2 = 2e^2 Q_i^2 g^2 \left\{ (1-\epsilon)\left(\frac{t}{u} + \frac{u}{t}\right) - 2\frac{sQ^2}{tu} - 2\epsilon \right\}$$



$$W_T = \frac{1}{4\pi M} \int_x^1 \frac{dy}{y} g(y) \tilde{W}_T$$

Scattering on a single gluon

Now we need the phase space

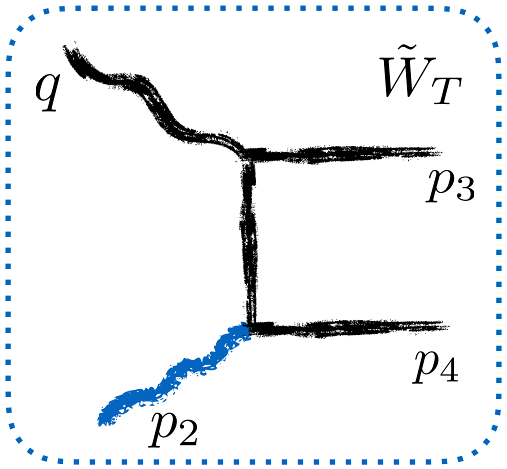
$$\int \frac{d^{d-1}p_3}{(2\pi)^{d-1} 2E_3} \frac{d^{d-1}k}{(2\pi)^{d-1} 2E_k} (2\pi)^d \delta^d(p_1 + p_2 - p_3 - k) = \frac{1}{8\pi} \left(\frac{4\pi}{s}\right)^{\epsilon} \left(1 - \frac{M^2}{s}\right)^{1-2\epsilon} \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dv (1-v)^{-\epsilon} v^{-\epsilon}$$

The result from the previous calculation

Now we integrate over two quarks in the final state and take into account that the quark is massless

$$\int \frac{d^{d-1}p_3}{(2\pi)^{d-1} 2E_3} \frac{d^{d-1}p_4}{(2\pi)^{d-1} 2E_4} (2\pi)^d \delta^d(q + p_2 - p_3 - p_4) = \frac{1}{8\pi} \left(\frac{4\pi}{s}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dv (1-v)^{-\epsilon} v^{-\epsilon}$$

Phase space



$$\frac{1}{16(1-\epsilon)} \sum_{\text{spin, color}} |M|^2 = 2e^2 Q_i^2 g^2 \left\{ (1-\epsilon)\left(\frac{t}{u} + \frac{u}{t}\right) - 2\frac{sQ^2}{tu} - 2\epsilon \right\}$$

Express Mandelstam variable in terms of angle

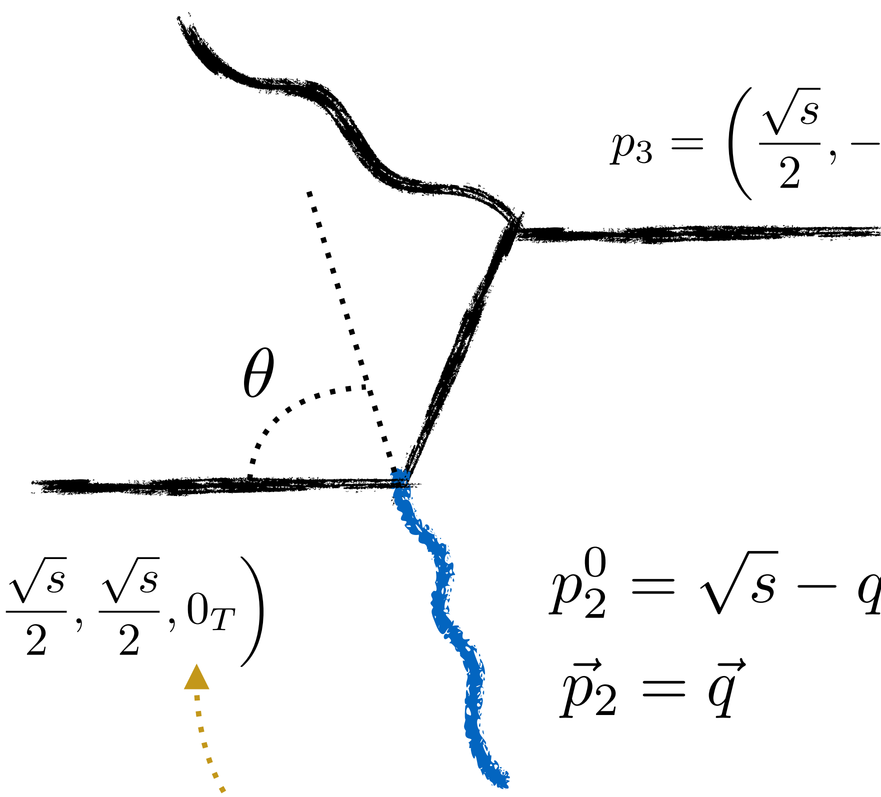
$$\int \frac{d^{d-1}p_3}{(2\pi)^{d-1}2E_3} \frac{d^{d-1}p_4}{(2\pi)^{d-1}2E_4} (2\pi)^d \delta^d(q + p_2 - p_3 - p_4) = \frac{1}{8\pi}\left(\frac{4\pi}{s}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dv (1-v)^{-\epsilon} v^{-\epsilon}$$

Take into account photon virtuality and get

$$t = -2p_2 \cdot p_4 = -s\left(1 + \frac{Q^2}{s}\right)(1-v)$$

$$u = -2p_2 \cdot p_3 = -s\left(1 + \frac{Q^2}{s}\right)v$$

We substitute this into square of the amplitude and perform integration



$$p_3 = \left(\frac{\sqrt{s}}{2}, -\frac{\sqrt{s}}{2}, 0_T\right)$$

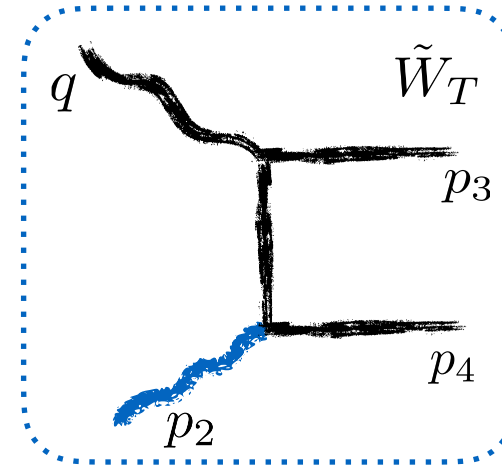
$$p_4 = \left(\frac{\sqrt{s}}{2}, \frac{\sqrt{s}}{2}, 0_T\right)$$

$$p_2^0 = \sqrt{s} - q^0$$

$$\vec{p}_2 = \vec{q}$$

d-2 dimensional vector

Transverse structure function



$$\tilde{W}_T = \frac{1}{16(1-\epsilon)} \sum_{\text{spin, color}} |M|^2 \times \frac{d^{d-1}p_3}{(2\pi)^{d-1}2E_3} \frac{d^{d-1}p_4}{(2\pi)^{d-1}2E_4} (2\pi)^d \delta^d(q+p_2-p_3-p_4)$$

There is no flux factor

$$W_T = \frac{1}{4\pi M} \int_x^1 \frac{dy}{y} g(y) \tilde{W}_T$$

Remove electron charge

$$\tilde{W}_T = 2Q_i^2 g^2 \frac{1}{8\pi} \left(\frac{4\pi}{s}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)} \int_0^1 dv (1-v)^{-\epsilon} v^{-\epsilon} \left\{(1-\epsilon)\left(\frac{1-v}{v} + \frac{v}{1-v}\right) - \frac{2Q^2}{s(1+Q^2/s)^2} \frac{1}{v(1-v)} - 2\epsilon\right\}$$

Kinematic variables

In the leading order z=1

$$x = \frac{Q^2}{2P\cdot q} = y\frac{Q^2}{2p_2 \cdot q} \qquad q^2 = -Q^2 \qquad s = 2p_2\cdot q - Q^2 \qquad z \equiv x/y$$

There is easy to calculate this integral using B function

$$\tilde{W}_T = 2Q_i^2 g^2 \frac{1}{8\pi} \left(\frac{4\pi}{Q^2}\frac{z}{1-z}\right)^{\epsilon} \frac{1}{\Gamma(1-\epsilon)}$$

$$\times \int_0^1 dv (1-v)^{-\epsilon} v^{-\epsilon} \left\{(1-\epsilon)\left(\frac{1-v}{v} + \frac{v}{1-v}\right) - 2z(1-z)\frac{1}{v(1-v)} - 2\epsilon\right\}$$